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Forced system with vibro-impact energy sink: chaotic strongly modulated responses.

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Abstract

The paper treats forced response of primary linear oscillator with vibro-impact energy sink. This system exhibits some features of dynamics, which resemble forced systems with other types of nonlinear energy sinks, such as steady-state and strongly modulated responses. However, the differences are crucial: in the system with vibro-impact sink the strongly modulated response consists of randomly distributed periods of resonant and non-resonant motion. This salient feature allows us to identify this type of dynamic behavior as chaotic strongly modulated response (CSMR). It is demonstrated, that the CSMR exists due to special structure of a slow invariant manifold (SIM), which is derived with the help of a multiple-scale analysis of the system. In the considered system, this manifold has only one stable and one unstable branch. This feature defines new class of universality for the nonlinear energy sinks. In the system with the vibro-impact sink, such responses are observed even for very low level of the external forcing. This feature makes such system viable for possible energy harvesting applications.

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Keywords: nonlinear energy sink; targeted energy transfer; vibro-impact energy sink; forced nonlinear system; strongly modulated response; energy harvesting

1. Introduction

Nonlinear energy sink (NES) is defined as relatively light essentially nonlinear attachment to a primary mechanical system[1-5]. Such attachments are capable of almost irreversible targeted energy transfer (TET) from the primary system to the NES, if the former is excited above certain energetic threshold[6,7]. Many recent studies were

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devoted to the TET in the systems comprising the NESs – from viewpoints of both general theory and numerous possible applications[8-15].

General definition of the NES presented above contains a notion of "essential nonlinearity". This term is somewhat vague, but usually "essentially nonlinear" should be understood as "non-linearizable". It means that the linear approximation is unable to capture essential features of the NES dynamics. Basic property of the essentially nonlinear system is that it can be excited in relatively broad range of frequencies; the latter strongly depend on energy stored in the system. Thus, the TET is possible due to a resonance between the primary system and the NES, which can "adjust" itself to the required frequency.

Theoretically, the type of nonlinearity involved in the NES may be rather diverse. In the first papers devoted to the subject[1-3,5] pure cubic nonlinearity was considered. Later more involved types of nonlinear stiffness were explored, including general non-polynomial[16] monotonic functions, NES with multiple states of equilibrium[17,18], as well as non-smooth and vibro-impact (VI) NES[11,19-22]. Nonlinear stiffness close to purely cubic was realized with the help of elastic strings or springs with minimal pre-tension[5]. The non-smooth NES was designed by combining linear elastic and vibro-impact elements[5]. Single-sided vibro-impact NES has been analyzed numerically and experimentally[23]. Recently, it was demonstrated that simple eccentric rotator can be efficiently used as the NES[24-26]. Two latter types of the NES (vibro – impact and rotational) are quite different from the other types, since they do not require any nonlinear spring. This drastic simplification makes them arguably the most viable candidates for diverse NES applications.

It should be mentioned that the vibro – impact elements were recognized and implemented long ago as viable engineering solutions for different problems related to vibration absorption and mitigation[27-35]. Commonly, analysis and design of such vibro – impact vibration absorbers considered only steady-state responses to external forcing with constant amplitude and frequency spectrum. The TET process is quite different, since it is the transient response of the system. Therefore the methods commonly used for the analysis of the steady – state responses are not sufficient.

To date, many attempts to describe the dynamics of the VI NES were concentrated around numeric simulations of conservative and damped dynamics[19-20]. For smooth NESs, approximate analytic description of transient damped responses has been achieved by singular perturbation approach, based on averaging and multiple scales analysis[5, 36]. Recently this method has been extended for the VI energy sinks in the case of impulsive loadings[37].

This paper is devoted to exploration of a forced primary linear system with vibro-impact energy sink. We consider the most popular case of direct harmonic forcing. Dynamic responses of forced systems including smooth NESs were widely studied in recent years[5, 38-40]. Important feature of these systems is a broad variety of possible response regimes. One can observe common steady-state responses with constant amplitude; these responses can lose stability through Neimark-Sacker bifurcation and convert into weakly modulated responses[40]. Besides, generically such systems possess strongly modulated responses (SMR). These regimes can be described as relaxation oscillations of the averaged dynamic flow[39-44]. Similar analysis of the forced system with VI energy sink requires certain modification of existing approaches and reveals some novel features.

2. Description of the model and analytic treatment

The model comprises primary forced-damped linear oscillator with the VI NES. The latter in this model is just a straight cavity in the primary mass, in which the impacting particle is allowed to move freely. Sketch of the system is presented in figure 1. The restitution coefficient is adopted to be less than unity: $\kappa < 1$. This is the only source of damping in the NES.

Without affecting the generality, we obtain non-dimensional model, as the primary mass and the rigidity of the linear spring are set to unity, and length of the cavity, in which the small mass moves, is equal to 2 (see Fig.1). The small mass of the NES is adopted to be equal to ε. The linear damping coefficient of the primary oscillator is $\varepsilon\xi$ The displacements of the primary mass and the impacting mass are denoted as $u(t)$ and $v(t)$, respectively.

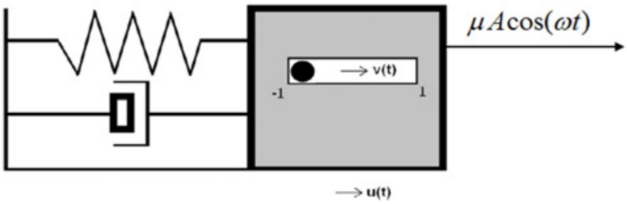


Fig. 1. Sketch of the dynamical system: primary forced linear oscillator with the VI NES

External forcing is supposed to be harmonic, with amplitude μA and frequency ω. Let us introduce a relative displacement of the NES with respect to the primary mass as

$$w(t) = u(t) - v(t) \tag{1}$$

The impacts occur when $w(t_j) = \pm 1$, where t_j is a time instance of the j^{th} impact. We adopt traditional Newtonian concept of an instantaneous impact, in which the relative velocity of the impacting elements is changed in accordance with the restitution coefficient:

$$\dot{w}(t_j + 0) = -\kappa \dot{w}(t_j - 0) \tag{2}$$

The impacts are instantaneous, and finite external forces do not modify total momentum in the course of the impact. Momentum conservation during each impact is expressed as:

$$\dot{u}(\mathrm{t}_j - 0) + \varepsilon \dot{v}(\mathrm{t}_j - 0) = \dot{u}(\mathrm{t}_j + 0) + \varepsilon \dot{v}(\mathrm{t}_j + 0) \tag{3}$$

General amount of momentum transferred to the primary mass in each impact is derived from (1-3):

$$\dot{u}(t_j + 0) - \dot{u}(t_j - 0) = -\frac{\varepsilon(1+\kappa)}{1+\varepsilon}\dot{w}(t_j - 0) \tag{4}$$

With the help of (4) one can write down general equations of motion of the model in terms of distributions. We take into account that the amount of momentum transferred to the primary mass is a sum of all the momenta transferred in all impacts. This system of equations is expressed as follows:

$$\begin{cases} \ddot{u} + \varepsilon\xi\dot{u} + u + \dfrac{\varepsilon\left(1+\kappa\right)}{1+\varepsilon}\sum\limits_j \dot{w}\left(t-0\right)\delta\left(t-t_j\right) = \mu A\cos(\omega t) \\ \ddot{v} - \dfrac{\left(1+\kappa\right)}{1+\varepsilon}\sum\limits_j \dot{w}\left(t-0\right)\delta\left(t-t_j\right) = 0 \end{cases} \tag{5}$$

As for the external forcing, we consider the most interesting case of a primary resonance. Consequently, the forcing frequency is close to that of the primary linear oscillator. In these conditions, the external forcing with small amplitude will cause significant effects. We adopt the following rescaling (the time is rescaled for convenience of notation):

$$\tau = \frac{t}{\sqrt{(1-\varepsilon\sigma)(1+\varepsilon)}}, \mu = \frac{\varepsilon}{(1-\varepsilon\sigma)(1+\varepsilon)}; \quad \omega = \frac{1}{\sqrt{(1-\varepsilon\sigma)(1+\varepsilon)}} \tag{6}$$

We further introduce a new displacement variable, proportional to the displacement of the center of masses of the system, and new damping coefficient as follows:

$$X(t) = u(t) + \varepsilon v(t), \gamma = \frac{\xi}{\sqrt{(1-\varepsilon\sigma)(1+\varepsilon)}} \tag{7}$$

Substituting (6) into (5) yields:

$$\begin{cases} X_{\tau\tau} + (1-\varepsilon\sigma)X + \varepsilon\gamma X_\tau + \varepsilon^2\gamma w_\tau + \varepsilon(1-\varepsilon\sigma)w = \varepsilon A cos(\tau) \\ \quad w_{\tau\tau} + (1-\varepsilon\sigma)X + \varepsilon\gamma X_\tau + \varepsilon^2\gamma w_\tau + \varepsilon(1-\varepsilon\sigma)w + \\ \quad +(1+\kappa)\sum_j \dot{w}(\tau-0)\delta(\tau-\tau_j) = \varepsilon A cos(\tau) \end{cases} \tag{8}$$

Now one can see that parameter σ quantifies small frequency detuning of the external force. System (8) is amenable to analysis with the help of multiple scales method, if some further natural assumptions are made. In further treatment, we suggest that the mass of the NES is small compared to the primary mass. All other values are adopted to be of order unity.

Multiple scales are introduced with the help of the following standard relationships :

$$\begin{gathered} \tau_k = \varepsilon^k\tau, k = 0,1\ldots, \qquad \frac{d}{dt} = \frac{\partial}{\partial\tau_0} + \varepsilon\frac{\partial}{\partial\tau_1}\ldots \\ X = X_0(\tau_0,\tau_1\ldots) + \varepsilon X_1(\tau_0,\tau_1\ldots) + \ldots \\ w = w_0(\tau_0,\tau_1\ldots) + \varepsilon w_1(\tau_0,\tau_1\ldots) + \ldots \end{gathered} \tag{9}$$

In current problem, only time scale τ_0 (referred to as fast time scale) and τ_1 (slow time scale) are required. Substituting (9) to (8) and keeping only ε^0-terms, we obtain the system of equations with respect to the fast time scale:

$$\begin{cases} \frac{\partial^2}{\partial\tau_0^2}X_0 + X_0 = 0 \\ \frac{\partial^2}{\partial\tau_0^2}w_0 + X_0 + (1+\kappa)\sum_j \frac{\partial w_0}{\partial\tau_0}\big|_{t_0-0}\ \delta(\tau_0-\tau_{0j}) = 0 \end{cases} \tag{10}$$

Solution of the first equation of system (10) is trivial:

$$X_0 = B(\tau_1)\sin(\tau_0 + \phi(\tau_1)) \tag{11}$$

Here $B(\tau_1), \phi(\tau_1)$ are amplitude and phase of X_0 respectively. They depend only on the slow time. Substituting (11) to the second equation of system (10) yields:

$$\frac{\partial^2}{\partial \tau_0^2} w_0 + (1+\kappa) \sum_j \frac{\partial w_0}{\partial \tau_0} |_{t_0 - 0} \delta(\tau_0 - \tau_{0j}) = -B(\tau_1) \sin(\tau_0 + \phi(\tau_1)) \tag{12}$$

Equation (12) describes a motion of forced particle with inelastic vibro-impact constraints. The amplitude and the phase of forcing in (12) should be considered as constants, since they depend only on the slow time. Solution of equation (12) is presented in the following form:

$$w_0 = X_0 + f(\tau_0, \tau_1) = B(\tau_1) \sin(\tau_0 + \phi(\tau_1)) + f(\tau_0, \tau_1) \tag{13}$$

Substituting (13) to (12), one obtains:

$$\frac{\partial^2 f}{\partial \tau_0^2} + (1+\kappa) \sum_j \left(\left. \frac{\partial f}{\partial \tau_0} \right|_{\tau_0 - 0} + B(\tau_1) \cos(\tau_0 + \phi(\tau_1)) \right) \delta(\tau_0 - \tau_{0j}) = 0 \tag{14}$$

The simplest case of the TET involves a capture into 1:1 resonance between the primary system and the NES. Then, in order to study the energy dissipation in the system, we look for the solution of equation (14) which corresponds to symmetric motion of the particle with the frequency of the external oscillations. Thus, the transient process leading to a capture into this stable response regime is not considered. Therefore, one can look for solution of equation (14) in the following form of the saw-tooth function[33-35]:

$$f(\tau_0, \tau_1) = \frac{2\alpha}{\pi} \arcsin(\cos(\tau_0 - \eta)) \tag{15}$$

Here α is the amplitude of the function *f*, and η is the phase shift between the external force and the impacts. One can observe from equation (15) that the impacts occur for $\tau_0 = \eta + n\pi$ where $n = 0, \pm 1, \pm 2...$. Integration of equation (15) over time in a small interval around $\tau_0 = \eta$ yields:

$$-\frac{4\alpha}{\pi} + (1+\kappa)\left(\frac{2\alpha}{\pi} + B\cos(\eta + \phi) \right) = 0 \tag{16}$$

From (16) we obtain the following relationship between the system parameters:

$$B\cos(\eta + \phi) = \rho\alpha \ ; \ \rho = \frac{2(1-\kappa)}{\pi(1+\kappa)} \tag{17}$$

As mentioned before, impacts occur when $w = \pm 1$ and $\tau_0 = \eta + n\pi$. Substituting (15) to (13), one obtains the second relationship between the system's parameters:

$$B\sin(\eta + \phi) + \alpha = 1 \tag{18}$$

It is necessary to keep in mind that α and η are functions of slow time scale τ_1. From (17) and (18) one obtains the following expression for $\alpha(B)$:

$$\alpha = \frac{1 \pm \sqrt{\rho^2+1}\sqrt{B^2 - B_{min}{}^2}}{\rho^2+1} \;; \quad B_{min} = \frac{\rho}{\sqrt{\rho^2+1}} \tag{19}$$

One can notice that in equation (19) there is an expression for B_{min} which is the minimum value for the amplitude of X_0. Equation (19) describes slow invariant manifold (SIM) of the system – set of fixed points for amplitude of oscillations for the fast time scale. This invariant manifold is composed from only one stable and one unstable branch, unlike the typical invariant manifold for NES system (even non-smooth) that has two stable branches[5, 36, 40, 45, 46]. Consequences of this difference are crucial and will be discussed later. The invariant manifold is presented in figure 2.

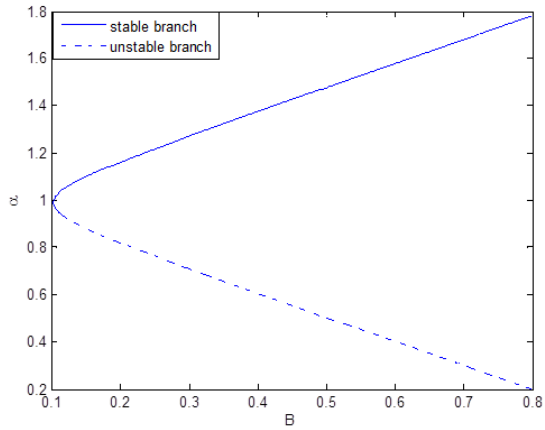


Figure 2. Slow invariant manifold for $\kappa = 0.729$

In order to understand the effect of forcing, we should consider the motion of the system at stable (upper) SIM branch, defined as $S^+ = \{B \in [B_{\min}, \infty), \phi \in [0, 2\pi)\}$. For this sake, we should consider the slow time scale in equation. (8). Keeping terms of order ε in the first equation of system (8), one obtains the following equation:

$$\frac{\partial^2 X_1}{\partial \tau_0^2} + 2\frac{\partial^2 X_0}{\partial \tau_0 \partial \tau_1} - \sigma X_0 + X_1 + \gamma \frac{\partial X_0}{\partial \tau_0} + X_0 + f(\tau_0, \tau_1) = A\cos(\tau_0) \tag{20}$$

Function $f(\tau_0, \tau_1)$ is then presented in a form of generalized Fourier series:

$$f(\tau_0, \tau_1) = \frac{2\alpha}{\pi}\arcsin\left(\cos\left(\tau_0 - \eta\right)\right) = \frac{8\alpha}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\cos\left((2k-1)(\tau_0 - \eta)\right) \tag{21}$$

Substituting (11) and (21) to (20), one finally obtains:

$$\begin{aligned}\frac{\partial^2 X_1}{\partial \tau_0^2} + X_1 = &-2\frac{\partial B}{\partial \tau_1}\cos(\tau_0+\phi) + 2B\frac{\partial \phi}{\partial \tau_1}\sin(\tau_0+\phi) - \gamma B\cos(\tau_0+\phi) - \\ &-(1-\sigma)B\sin(\tau_0+\phi) + A\cos(\tau_0+\phi)\cos(\phi) + A\sin(\tau_0+\phi)\sin(\phi) - \\ &-\frac{8\alpha}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\left(\cos(\tau_0+\phi)\cos(\eta+\phi) + \sin(\tau_0+\phi)\sin(\eta+\phi)\right)\end{aligned} \tag{22}$$

Evolution equations for the slow scale for $B(\tau_1)$ and $\eta(\tau_1)$ are now obtained from a condition of absence of secular terms in (22). With the help of (16) and (17), one arrives to the following result:

$$\begin{cases} \dfrac{\partial B}{\partial \tau_1}+\dfrac{\gamma B}{2}+\dfrac{4\rho\alpha^2}{\pi^2}=\dfrac{A}{2}\cos(\phi) \\ B\dfrac{\partial \phi}{\partial \tau_1}-(1-\sigma)\dfrac{B}{2}-\dfrac{4\alpha}{\pi^2}(1-\alpha)=-\dfrac{A}{2}\sin(\phi) \end{cases} \tag{23}$$

From equations (23) one can see that for A small enough there are no steady-state solutions (fixed points at the stable SIM branch). It is also obvious from physical reasons – too small forcing is unable to counterweight the dissipation, and, the steady-state solution cannot be supported. From the other side, it is obvious that if A is large enough, at least one stable stationary point should appear. The phase portrait of the slow flow, which corresponds to the case where the stationary points do not exist (low forcing amplitude), is presented in figure 3.

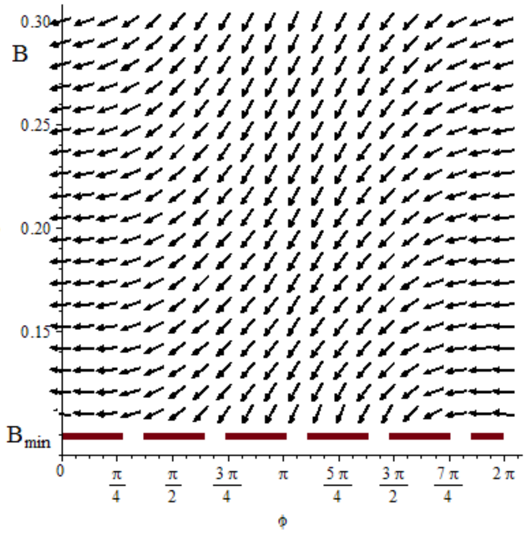


Fig. 3. Phase portrait for the stable SIM branch for the case of weak forcing, $\sigma = 1, \kappa = 0.729, \gamma = 0, A = 0.1$

Qualitative shape of the phase flow in Fig. 3 allows one to describe the resonant response regimes of the considered system. It is obvious that eventually the phase point attracted to the SIM should "jump" out from the SIM. Main question here is – where should it "land" after the jump? In traditional picture of the relaxation oscillations, the SIM is S-shaped and the phase point "lands" at the lower stable SIM branch. This mechanism is realized in many systems including NES and leads, in particular, to stable strongly modulated responses (SMR)[5, 40]. However, in the explored vibro-impact NES the situation turns out to be completely different – the SIM does not have lower stable branch. The only possible scenario is that after the "jump" from the fold the system leaves the regime of 1:1 resonance. In other words, the dissipation related with the impact motion of the NES almost ceases to suppress the oscillations of the primary mass. As a result, the primary oscillator is energized again and the amplitude of the primary mass increases. Then the system can be captured again by the SIM and again will move towards the fold, dissipating the energy of vibrations. So, one obtains an intermittent captures into 1:1 resonance manifold, divided by intervals of the non-resonant motion.

This non-resonant motion is accompanied by impacts, which are not synchronized with the external excitation. One would expect that such motion would strongly depend on particular initial conditions. Consequently, the point of the next capture by the SIM branch will also sensitively depend on the point of the fold, at which the previous capture commences by the “jump”, and on the phase of external force. Therefore, one can predict that the flow will be eventually captured by the stable SIM branch, but cannot predict at which point it will happen and how long will it take. Evolution of such flow on the SIM and beyond is schematically illustrated in figure 4.

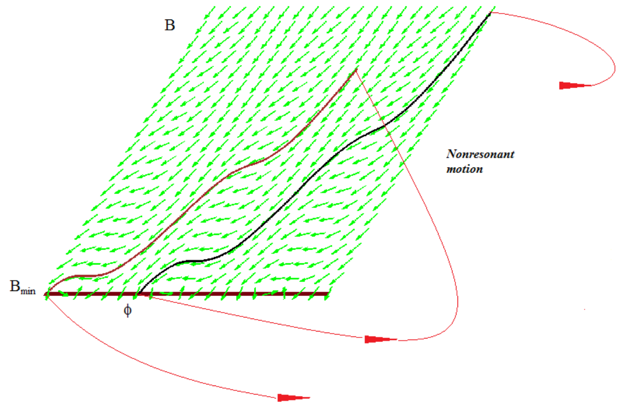
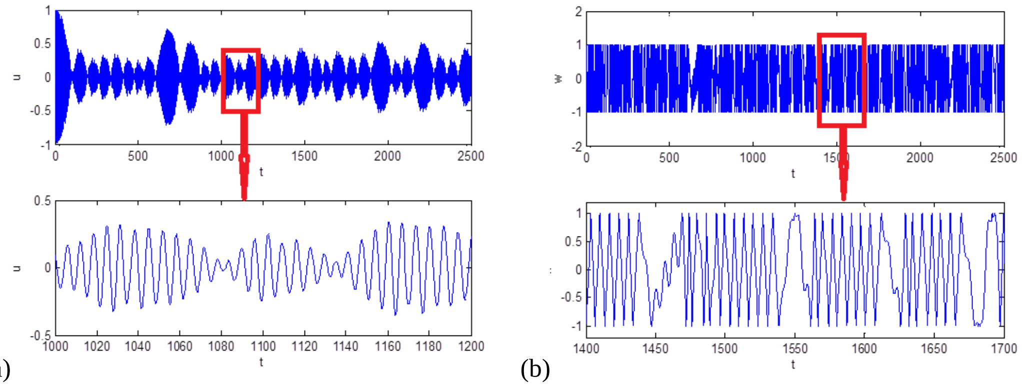


Figure 4. Scheme of chaotic resonance captures on the SIM.

As a result, one can expect very special dynamics: “bursts” of modulated 1:1 oscillations with random length and amplitude, divided by intervals of irregular non-resonant motion. To the best of the authors’ knowledge, such response regime was never described before. Similarly to traditional SMR, it is related to separation of time scales and appearance of the SIM. Due to special topology of the SIM in our problem (non-existence of the lower stable branch), this SMR should exhibit profound chaotic properties. We’ll refer to it as chaotic strongly modulated response (CSMR).

3. Numeric exploration and verification

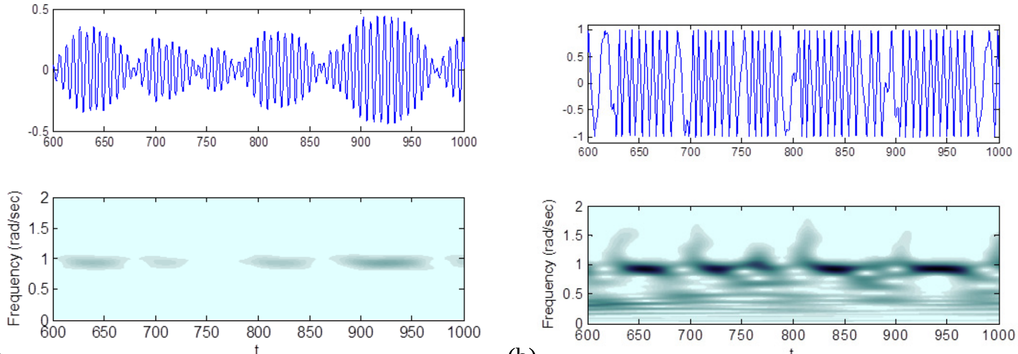
In figures 5a,b the numeric verification for the CSMR for the case of the forcing amplitude *A=0.5* is presented. Exact values of all simulation parameters are presented in the figure captions.



(a) (b)

Fig. 5. Chaotic strongly modulated response for the case of intermediate forcing amplitude, $\varepsilon = 0.05, \sigma = 1, \xi = 0.2, \kappa = 0.729, A = 0.5$. (a) Displacement of the primary system; (b) Relative displacement of the NES.

In figure 5b it is easy to distinguish relatively long periods of resonant motion, divided by apparently irregular non-resonant zones. Further validation of this scenario can be obtained from wavelet analysis of these time series, presented in figure 6.



(a) (b)

Fig, 6. Wavelet decomposition of the time series for CSMR, $\varepsilon = 0.05, \sigma = 1, \xi = 0.2, \kappa = 0.729, A = 0.5$. (a) Displacement of the primary system; (b) Relative displacement of the NES.

Figure 6b confirms that at moderate forcing amplitude (*A=0.5*) for majority of time the vibro-impact NES is engaged into 1:1 resonance. Between these captures one observes a broad-band response, which supports to some extent the conjecture on the chaotic character of the observed process. To get direct evidence of chaotic behavior, we compute numerically Lyapunov spectrum of the model system. The result is presented in figure 7.

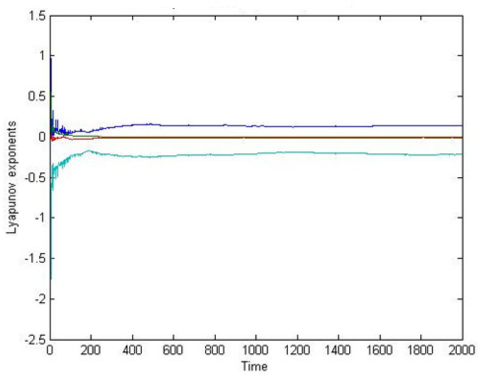


Fig. 7. Lyapunov exponents versus time for model system with zero initial conditions and parameters $\varepsilon = 0.05, \sigma = 1, \xi = 0.2, \kappa = 0.729, A = 0.5$.

One Lyapunov exponent is positive, and therefore the CSMR response is indeed chaotic.

4. Discussion and concluding remarks

Analytic and numeric results presented in this paper add new arguments to already existing perception, that structure and generic properties of forced responses in various systems with nonlinear energy sinks are determined by dimensionality and structure of the slow invariant manifolds; the latter reveal themselves in the process of multiple-scale decomposition[40]. These SIMs are sets of fixed points for “the second slowest” time scale involved in the dynamics – fast time scale for the considered problem, or slow time scale for the smooth NESs[5, 40]. The slowest time scale (slow time in current problem and super-slow for the smooth NES) describes, in turn, dynamics of the system on the SIM.

So, one can predict that the CSMR response should appear also in other forced systems with the NES, in which the SIM has the structure similar to one presented in figure 3 – with only one stable and one unstable branch. Such structure of the SIM is known for other NES type, very different from one described above – the rotational NES[24-27].

Without going into much detail, we present here a result of simulations for forced linear oscillator with rotational NES. Equations of motion in non-dimensional form are presented below:

$$\ddot{u} = \frac{-(1-\varepsilon\sigma)u + \varepsilon(\dot{\theta}^2\cos\theta + \gamma_2\dot{\theta}\sin\theta - \gamma_1\dot{u} + A\cos t)}{1-\varepsilon\sin^2\theta};$$
$$\ddot{\theta} = \frac{-\gamma_2\dot{\theta} + \sin\theta(-(1-\varepsilon\sigma)u + \varepsilon(\dot{\theta}^2\cos\theta - \gamma_1\dot{u} + A\cos t))}{1-\varepsilon\sin^2\theta}. \quad (24)$$

Here $\theta(t)$ is a rotation angle of the NES with respect to the direction of motion of the primary mass, $\varepsilon\gamma_1$ and $\varepsilon\gamma_2$ are linear damping coefficient for the primary mass and the NES respectively. Other coefficients are similar to those defined above in Sect. 2.

Details of derivation of system (24) (without external forcing) can be found in[24-27]. CSMR regime in this system is presented in figure 8.

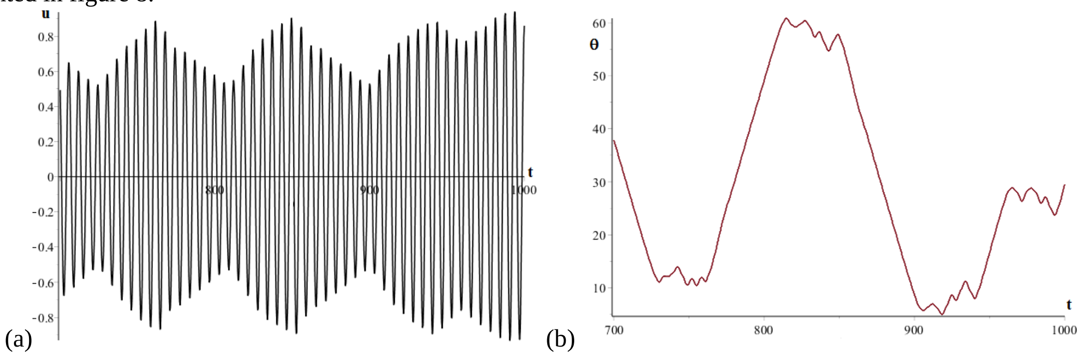


Fig. 8. CSMR response for the forced system with rotational NES, $\varepsilon = 0.05, \sigma = 0, \gamma_1 = 0.6, \gamma_2 = 0.3, A = 1$. All initial conditions are zero, besides $\theta(0) = 0.1$. a) Displacement of the primary oscillator; b) Angle of the NES rotation.

It is interesting to point on one important feature of the CSMR in the system with VI NES. This response regime is observed for very low values of the external forcing. In this case the primary system is excited for relatively long time, and then rapidly releases energy through the resonance with the NES. Such property is in a sense unique for the VI NES, since the smooth NESs normally can possess a fixed point on the lower stable branch of the SIM. Such property of the system might be useful for energy harvesting applications; even small ambient forcing will bring about relatively large displacements of the primary system, and the energy could be released and harvested through short pulses related to the NES activation.

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